

# AROUND OPERATOR MONOTONE FUNCTIONS

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**ABSTRACT.** We show that the symmetrized product  $AB + BA$  of two positive operators  $A$  and  $B$  is positive if and only if  $f(A + B) \leq f(A) + f(B)$  for all non-negative operator monotone functions  $f$  on  $[0, \infty)$  and deduce an operator inequality. We also give a necessary and sufficient condition for that the composition  $f \circ g$  of an operator convex function  $f$  on  $[0, \infty)$  and a non-negative operator monotone function  $g$  on an interval  $(a, b)$  is operator monotone and give some applications.

## 1. INTRODUCTION

Let  $\mathbb{B}(\mathcal{H})$  denote the algebra of all bounded linear operators on a complex Hilbert space  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ . An operator  $A \in \mathbb{B}(\mathcal{H})$  is called *positive* if  $\langle Ax, x \rangle \geq 0$  holds for every  $x \in \mathcal{H}$  and then we write  $A \geq 0$ . The set of all positive operators on  $\mathcal{H}$  is denoted by  $\mathbb{B}(\mathcal{H})_+$ . For self-adjoint operators  $A, B \in \mathbb{B}(\mathcal{H})$ , we say  $A \leq B$  if  $B - A \geq 0$ . The symmetrized product of two operators  $A, B \in \mathbb{B}(\mathcal{H})$  is defined by  $S(A, B) = AB + BA$ . In general, the symmetrized product of two operators  $A, B \in \mathbb{B}(\mathcal{H})_+$  is not positive. For example, if  $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ , then  $AB + BA$  is not a positive operator. The operator  $\frac{1}{2}S(A, B)$  is called the *Jordan product* of  $A$  and  $B$ . Nicholson [11] and Strang [12] introduced some sufficient conditions for that the Jordan product of two positive matrices  $A$  and  $B$  is positive. Also Gustafson [7] showed that if  $0 \leq m \leq A \leq M$  and  $0 \leq n \leq B \leq N$ , then

$$mn - \frac{(M - m)(N - n)}{8} \leq \frac{1}{2}S(A, B). \quad (1)$$

Throughout the paper we assume all functions to be continuous. Let  $f$  be a real-valued function defined on an interval  $J$ . If for each self-adjoint operators  $A, B \in \mathbb{B}(\mathcal{H})$  with spectra in  $J$ ,

- $A \leq B$  implies  $f(A) \leq f(B)$ , then  $f$  is called *operator monotone*;

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- $f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B)$  for all  $\lambda \in [0, 1]$ , then  $f$  is said to be *operator convex*;
- $f(A + B) \leq f(A) + f(B)$ , then  $f$  is called *subadditive*.

If  $f$  is an operator monotone function on  $[0, \infty)$ , then  $f$  can be represented as

$$f(t) = f(0) + \beta t + \int_0^\infty \frac{\lambda t}{\lambda + t} d\mu(\lambda),$$

where  $\beta \geq 0$  and  $\mu$  is a positive measure on  $[0, \infty)$  and if  $f$  is an operator convex function on  $[0, \infty)$ , then  $f$  can be represented as

$$f(t) = f(0) + \beta t + \gamma t^2 + \int_0^\infty \frac{\lambda t^2}{\lambda + t} d\mu(\lambda),$$

where  $\gamma \geq 0$ ,  $\beta = f'_+(0) = \lim_{t \rightarrow 0^+} \frac{f(t) - f(0)}{t}$  and  $\mu$  is a positive measure on  $[0, \infty)$ ; see [5, Chapter V]. If  $f$  is a non-negative operator monotone function on  $[0, \infty)$ , then the subadditivity of  $f$  does not hold in general. Aujla and Bourin [2] showed that if  $A, B \geq 0$  are matrices and  $f : [0, \infty) \rightarrow [0, \infty)$  is a concave function, i.e.  $-f$  is convex, then there exist unitaries  $U, V$  such that

$$f(A + B) \leq Uf(A)U^* + Vf(B)V^*.$$

Regarding the subadditivity property, Ando and Zhan [1] proved that  $|||f(A + B)||| \leq |||f(A) + f(B)|||$  for all non-negative operator monotone functions  $f$ , all unitarily invariant norms  $||| \cdot |||$  and all matrices  $A, B \geq 0$ ; see also [3]. Recall that a norm  $||| \cdot |||$  on the algebra of all  $n \times n$  matrices is unitarily invariant if  $|||X||| = |||UXV|||$  for all unitaries  $U$  and  $V$  and all matrices  $X$ .

Throughout this paper  $\mathcal{M}(\mathcal{H})$  denotes the set of all  $(A, B) \in \mathbb{B}(\mathcal{H})_+ \times \mathbb{B}(\mathcal{H})_+$  for which  $f(A + B) \leq f(A) + f(B)$  for all non-negative operator monotone functions  $f$  on  $[0, \infty)$ . We shall show that

$$\mathcal{M}(\mathcal{H}) = \{(A, B) \in \mathbb{B}(\mathcal{H})_+ \times \mathbb{B}(\mathcal{H})_+ \mid AB + BA \geq 0\}.$$

We apply this result to present an operator inequality involving operator monotone functions analogue to that of Hansen [9]. We also give a necessary and sufficient condition for the composition  $f \circ g$  of an operator convex function  $f$  on  $[0, \infty)$  and a non-negative operator monotone function  $g$  on an interval  $(a, b)$  to be operator monotone and deduce some operator inequalities.

## 2. THE RESULTS

We start this section with one of our main results.

**Theorem 2.1.** *Let  $A, B \in \mathbb{B}(\mathcal{H})_+$ . Then  $AB+BA$  is positive if and only if  $f(A+B) \leq f(A) + f(B)$  for all non-negative operator monotone functions  $f$  on  $[0, \infty)$ .*

*Proof.* Suppose that  $f(A+B) \leq f(A) + f(B)$  for all non-negative operator monotone functions  $f$  on  $[0, \infty)$ . Let  $\lambda \in (0, \infty)$ . The function  $f_\lambda(t) = \frac{\lambda t}{\lambda+t}$  is operator monotone on  $[0, \infty)$ . Hence

$$(A+B)(\lambda+A+B)^{-1} \leq A(\lambda+A)^{-1} + B(\lambda+B)^{-1}. \quad (2)$$

Put  $X_\lambda = A(\lambda+A)^{-1}$  and  $Y_\lambda = B(\lambda+B)^{-1}$ . Inequality (2) is equivalent to

$$\begin{aligned} (\lambda+A+B)(A+B) &\leq (\lambda+A+B)X_\lambda(\lambda+A+B) \\ &\quad + (\lambda+A+B)Y_\lambda(\lambda+A+B) \\ &= \lambda^2 X_\lambda + 2\lambda A X_\lambda + \lambda B X_\lambda + \lambda X_\lambda B + A X_\lambda A + A X_\lambda B \\ &\quad + B X_\lambda A + B X_\lambda B + \lambda^2 Y_\lambda + 2\lambda B Y_\lambda + \lambda A Y_\lambda \\ &\quad + \lambda Y_\lambda A + B Y_\lambda B + B Y_\lambda A + A Y_\lambda B + A Y_\lambda A. \end{aligned}$$

Since  $BY_\lambda = Y_\lambda B = B - \lambda Y_\lambda$  and  $AX_\lambda = X_\lambda A = A - \lambda X_\lambda$  the above inequality is, in turn, equivalent to

$$\begin{aligned} \lambda(A+B) + A^2 + B^2 + AB + BA &\leq \lambda(A+B) + A^2 + B^2 \\ &\quad + 2(AB + BA) + B X_\lambda B + A Y_\lambda A \end{aligned}$$

or

$$AB + BA + B X_\lambda B + A Y_\lambda A \geq 0. \quad (3)$$

Letting  $\lambda \rightarrow \infty$  we get  $B X_\lambda B + A Y_\lambda A \rightarrow 0$ , whence  $AB + BA \geq 0$ .

Conversely, assume that  $AB + BA \geq 0$ . Since inequality (3) holds for each  $\lambda \in \mathbb{R}_+$ , we obtain  $f_\lambda(A+B) \leq f_\lambda(A) + f_\lambda(B)$ . Now, if  $f$  be a non-negative operator monotone function on  $[0, \infty)$ , then  $f$  can be represented on  $[0, \infty)$  by

$$f(t) = f(0) + \beta t + \int_0^\infty f_\lambda(t) d\mu(\lambda),$$

where  $\beta \geq 0$  and  $\mu$  is a positive measure on  $[0, \infty)$ . Since  $f(0) \geq 0$ , without loss of generality, we can assume that  $f(t) = \int_0^\infty f_\lambda(t) d\mu(\lambda)$ . Therefore

$$\begin{aligned} f(A+B) &= \int_0^\infty f_\lambda(A+B) d\mu(\lambda) \\ &\leq \int_0^\infty (f_\lambda(A) + f_\lambda(B)) d\mu(\lambda) \\ &= \int_0^\infty f_\lambda(A) d\mu(\lambda) + \int_0^\infty f_\lambda(B) d\mu(\lambda) \\ &= f(A) + f(B). \end{aligned}$$

□

If  $A, B \in \mathbb{B}(\mathcal{H})_+$  commute, then  $AB = (A^{1/2}B^{1/2})^2 \geq 0$ . The above theorem therefore shows that  $f(A+B) \leq f(A) + f(B)$  for any non-negative operator monotone function  $f$  on  $[0, \infty)$ . In particular, for each  $n \in \mathbb{N}$  we have  $f(nA) \leq nf(A)$ .

**Corollary 2.2.** *Let  $0 \leq m \leq A \leq M$ ,  $0 \leq n \leq B \leq N$  and  $f$  be a non-negative operator monotone function on  $[0, \infty)$ . If  $(M-m)(N-n) \leq 8mn$ , then*

$$f(A+B) \leq f(A) + f(B).$$

*Proof.* Use inequality (1). □

**Corollary 2.3.** *Let  $0 \leq p \leq \frac{1}{2}$  and  $f$  be a non-negative operator monotone function on  $[0, \infty)$ . Then for positive operators  $A$  and  $B$  with  $A \leq B$  it holds that*

$$f(B^p) \leq f\left(\frac{B^p + A^p}{2}\right) + f\left(\frac{B^p - A^p}{2}\right).$$

*Proof.* Assume that  $S_1 = \frac{1}{2}(B^p + A^p)$  and  $S_2 = \frac{1}{2}(B^p - A^p)$ . Clearly  $S_1 \geq 0$  and due to  $f(t) = t^p$  is operator monotone [6, Theorem 1.8],  $S_2 \geq 0$ . We have

$$2(S_1 S_2 + S_2 S_1) = (S_1 + S_2)^2 - (S_1 - S_2)^2 = B^{2p} - A^{2p} \geq 0,$$

since  $f(t) = t^{2p}$  is also operator monotone. Therefore

$$f(B^p) = f(S_1 + S_2) \leq f(S_1) + f(S_2) = f\left(\frac{B^p + A^p}{2}\right) + f\left(\frac{B^p - A^p}{2}\right).$$

□

Let  $m \leq M \leq (2\sqrt{2} + 1)m$ . If  $0 \leq m \leq A, B \leq M$ , then inequality (1) ensures that  $AB + BA$  is positive. From this we can introduce an inequality for non-negative operator monotone functions. We should notice that Hansen [9] proved that an operator monotone function  $f$  on  $[0, \infty)$  satisfies the inequality  $C^* f(A) C \leq f(C^* A C)$  for all

positive operators  $A$  and all contractions  $C$ , i.e. operators of norm less than or equal to one.

**Theorem 2.4.** *Let  $f$  be a non-negative operator monotone function on  $[0, \infty)$ . Let  $A$  and  $A_i$  ( $1 \leq i \leq n$ ) be positive operators with spectra in  $[\lambda, (1 + 2\sqrt{2})\lambda]$  for some  $\lambda \in \mathbb{R}_+$ . Then*

(i) *For every isometry  $C \in \mathbb{B}(\mathcal{H})$ ,*

$$f(C^*AC) \leq 2 C^* f\left(\frac{A}{2}\right) C.$$

(ii) *For operators  $C_i$  ( $i = 1, \dots, n$ ) with  $\sum_{i=1}^n C_i^* C_i = I$ ,*

$$f\left(\sum_{i=1}^n C_i^* A_i C_i\right) \leq 2 \sum_{i=1}^n C_i^* f\left(\frac{A}{2}\right) C_i.$$

*Proof.* (i) Put  $X = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ ,  $D = (I - CC^*)^{\frac{1}{2}}$ ,  $V = \begin{pmatrix} C & -D \\ 0 & C^* \end{pmatrix}$  and  $U = \begin{pmatrix} C & D \\ 0 & -C^* \end{pmatrix}$ . It is easy to see that  $U$  and  $V$  are unitary operators. So  $\text{sp}(U^* X U) = \text{sp}(V^* X V) = \text{sp}(X) \subseteq [\lambda, (1 + 2\sqrt{2})\lambda]$ , where “sp” stands for spectrum. Thus

$$\begin{aligned} \begin{pmatrix} f(C^*AC) & 0 \\ 0 & f(DAD + CAC^*) \end{pmatrix} &= f\begin{pmatrix} C^*AC & 0 \\ 0 & DAD + CAC^* \end{pmatrix} \\ &= f\left(\frac{U^* X U + V^* X V}{2}\right) \\ &\leq f\left(U^* \frac{X}{2} U\right) + f\left(V^* \frac{X}{2} V\right) \\ &= U^* f\left(\frac{X}{2}\right) U + V^* f\left(\frac{X}{2}\right) V \\ &= 2 \begin{pmatrix} C^* f\left(\frac{A}{2}\right) C & 0 \\ 0 & D f\left(\frac{A}{2}\right) D + C f\left(\frac{A}{2}\right) C^* \end{pmatrix}, \end{aligned}$$

whence  $f(C^*AC) \leq 2C^* f\left(\frac{A}{2}\right) C$ .

(ii) Set

$$\tilde{C} = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} A_1 & & 0 \\ & A_2 & \\ & & \ddots \\ 0 & & & A_n \end{pmatrix}.$$

Thus

$$f\left(\sum_{i=1}^n C_i^* A_i C_i\right) = f(\tilde{C}^* \tilde{A} \tilde{C}) \leq 2\tilde{C}^* f\left(\frac{\tilde{A}}{2}\right) \tilde{C} = 2 \sum_{i=1}^n C_i^* f\left(\frac{A_i}{2}\right) C_i.$$

□

**Corollary 2.5.** *Let  $\lambda \in \mathbb{R}_+$  and  $w_i \in [\lambda, (1 + 2\sqrt{2})\lambda]$  ( $i = 1, \dots, n$ ). Let  $f$  be a non-negative operator monotone function on  $[0, \infty)$  and  $A_i$  be positive operators such that  $\sum_{i=1}^n A_i = I$ . Then*

$$f\left(\sum_{i=1}^n w_i A_i\right) \leq 2 \sum_{i=1}^n f\left(\frac{w_i}{2}\right) A_i.$$

*Proof.* Put  $C_i = A_i^{\frac{1}{2}}$  in part (ii) of Theorem 2.4. □

**Theorem 2.6.** *Let  $A, B \in \mathbb{B}(\mathcal{H})_+$ . Then  $B^2 \leq A^2$  if and only if for each operator convex function  $f$  on  $[0, \infty)$  with  $f'_+(0) \geq 0$  it holds that*

$$f(B) \leq f(A)$$

*Proof.* Let  $A, B \in \mathbb{B}(\mathcal{H})_+$ . Suppose that for each operator convex function  $f$  on  $[0, \infty)$  with  $f'_+(0) \geq 0$  we have  $f(B) \leq f(A)$ . Due to  $f(t) = t^2$  is operator convex on  $[0, \infty)$  and  $f'_+(0) \geq 0$ , we get  $B^2 \leq A^2$ .

Next we show the converse. First assume that  $A$  and  $B$  are invertible positive operators such that  $B^2 \leq A^2$ . Then  $\lambda A^{-2} + A^{-1} \leq \lambda B^{-2} + B^{-1}$ . This inequality is equivalent to  $(\lambda B^{-2} + B^{-1})^{-1} \leq (\lambda A^{-2} + A^{-1})^{-1}$  and this is, in turn, equivalent to  $B^2(B + \lambda)^{-1} \leq A^2(A + \lambda)^{-1}$ . On the other hand  $f_\lambda(t) = \frac{\lambda t^2}{\lambda + t}$  is operator convex on  $[0, \infty)$ , so  $f_\lambda(B) \leq f_\lambda(A)$ .

If  $A$  and  $B$  are positive and  $(A - B, A + B) \in \mathcal{M}(\mathcal{H})$ , then for each  $\epsilon \geq 0$  let us set  $A_\epsilon = A + \epsilon$  and  $B_\epsilon = B + \epsilon$ . Then  $(A_\epsilon - B_\epsilon, A_\epsilon + B_\epsilon) \in \mathcal{M}(\mathcal{H})$ . Hence

$$f_\lambda(B + \epsilon) \leq f_\lambda(A + \epsilon).$$

Letting  $\epsilon \rightarrow 0$  we get  $f_\lambda(B) \leq f_\lambda(A)$ .

Now suppose that  $f$  is an operator convex function on  $[0, \infty)$  with  $f'_+(0) \geq 0$ . It is known that  $f$  can be represented on  $[0, \infty)$  by

$$f(t) = f(0) + \beta t + \gamma t^2 + \int_0^\infty f_\lambda(t) d\mu(\lambda), \quad (4)$$

where  $\gamma \geq 0$ ,  $\beta = f'_+(0)$  and  $\mu$  is a positive measure on  $[0, \infty)$ ; see [5, Chapter V]. If

$$f(t) = \int_0^\infty f_\lambda(t) d\mu(\lambda),$$

then due to  $f_\lambda(B) \leq f_\lambda(A)$  for each  $\lambda \in \mathbb{R}_+$  we have

$$f(B) \leq f(A).$$

Since  $\gamma, \beta \geq 0$  and  $B^2 \leq A^2$  the validity of  $f(B) \leq f(A)$  is deduced in the general case when  $f$  is given by (4).  $\square$

*Remark 2.7.* It follows from the identity

$$(A - B)(A + B) + (A + B)(A - B) = 2(A^2 - B^2)$$

that  $B^2 \leq A^2$  if and only if  $(A - B, A + B) \in \mathcal{M}(\mathcal{H})$ .

We need the Löwner Theorem for establishing our next main result.

**Theorem 2.8.** [10, Löwner Theorem] *A function  $g$  defined on  $(a, b)$  is operator monotone if and only if it is analytic in  $(a, b)$ , can analytically be continued to the whole upper half-plane  $\{z \in \mathbb{C} : \text{Im}(z) > 0\}$  and represents there an analytic function whose imaginary part is non-negative.*

**Theorem 2.9.** *Let  $g$  be a non-negative operator monotone function on an interval  $(a, b)$ . Let  $g(z) = u(z) + iv(z)$  be its analytic continuation to the upper half-plane. Then for each operator convex function  $f$  on  $[0, \infty)$  with  $f'_+(0) \geq 0$ ,  $f \circ g$  is operator monotone on  $(a, b)$  if and only if  $u(z) \geq 0$  on the upper half-plane.*

*Proof.* Assume that  $u(z) \geq 0$ . By the Löwner Theorem 2.8,  $v(z) \geq 0$  on the upper half-plane. One then sees that  $g$  maps the upper half-plane into the first quadrant of plane. Hence  $g^2$  maps the upper half-plane into itself. Utilizing again the Löwner Theorem we conclude that  $g^2$  is operator monotone. Assume that  $A$  and  $B$  are self-adjoint operators with spectra in  $(a, b)$  and  $B \leq A$ . It follows from  $(g(A) - g(B))(g(A) + g(B)) + (g(A) + g(B))(g(A) - g(B)) = 2(g(A)^2 - g(B)^2)$  that  $(g(A) - g(B), g(A) + g(B)) \in \mathcal{M}(\mathcal{H})$ . Now Theorem 2.6 implies that for each operator convex function  $f$  on  $[0, \infty)$  with  $f'_+(0) \geq 0$  the function  $f \circ g$  is operator monotone on  $(a, b)$ .

Conversely, assume that there would be a complex number  $z_0$  with  $\text{Im} z_0 > 0$  such that  $u(z_0) < 0$ . Because  $g$  is operator monotone,  $v(z_0) \geq 0$ . Hence  $\frac{\pi}{2} < \text{Arg}(g(z_0)) < \pi$ . Therefore  $\pi < \text{Arg}(g^2(z_0)) < 2\pi$ . So that  $\text{Im } g^2(z_0) < 0$ . Using the Löwner Theorem 2.8,  $g^2$  is not operator monotone on  $(a, b)$ , which is a contradiction, since  $f(x) = x^2$  is operator convex on  $(0, \infty)$ .  $\square$

*Remark 2.10.* If a non-negative operator monotone function  $g$  on an interval  $(a, b)$  is not a zero constant function and  $u(z) \geq 0$  on the domain of  $g$ , then  $u(z) > 0$  on this domain. To see this note that:

- (i) If for some  $t_0 \in (a, b)$  we have  $g(t_0) = 0$ , then there exists  $R > 0$  such that  $g$  can be represented as  $g(t) = \sum_{n=1}^{\infty} \frac{g^{(n)}(t_0)}{n!} (t - t_0)^n$  for all  $t \in (t_0 - R, t_0 + R)$  [4, p. 63]. Since  $g$  is a non-negative monotone function,  $g(t) = 0$  for all  $t \leq t_0$ . Hence  $g^{(n)}(t_0) = 0$  for all  $n$ , so  $g$  is zero on the neighborhood  $(t_0 - R, t_0 + R)$  of  $t_0$ . Thus  $\{t : g(t) = 0\}$  is clopen. Hence  $g$  is zero on  $(a, b)$  contradicting the assumption above. Thus  $g(t) > 0$  on  $(a, b)$ .
- (ii) If  $g \neq 0$  is a constant function, then clearly  $u(z) > 0$ .
- (iii) If  $g$  is not a constant function, then by the open mapping theorem for non-constant analytic functions,  $u$  maps the upper half-plane into  $\{z : \operatorname{Im} z > 0 \text{ \& } u(z) > 0\}$ .

**Corollary 2.11.** *Let  $0 \leq p \leq \frac{1}{2}$  and let  $f$  be an operator convex function on  $[0, \infty)$  with  $f'_+(0) \geq 0$ . If  $B \leq A$ , then*

$$f(B^p) \leq f(A^p).$$

*Proof.* Since for  $0 \leq p \leq \frac{1}{2}$  the function  $g(t) = t^p$  is non-negative operator monotone and  $g$  takes the upper half-plane into the first quarter of plane, by Theorem 2.9 we get  $f \circ g$  is operator monotone on  $[0, \infty)$ .  $\square$

**Corollary 2.12.** *Let  $0 \leq p \leq \frac{1}{2}$  and  $f$  be a non-negative operator monotone function on  $[0, \infty)$ . Then for positive operators  $A$  and  $B$  with  $B \leq A$ ,*

- (i)  $B^p f(B^p) \leq A^p f(A^p)$ ;
- (ii) *If  $f$  is strictly positive on  $(0, \infty)$  and  $A, B$  are invertible, then  $A^{p-1} f(A^p) \leq B^{p-1} f(B^p)$ .*

*Proof.* (i) Due to  $f$  is operator monotone on  $[0, \infty)$ , by [8, Theorem 2.4], the function  $g(t) = t f(t)$  is operator convex on  $[0, \infty)$ , hence  $g'_+(0) = f(0) \geq 0$ . Corollary 2.11 then yields

$$B^p f(B^p) \leq A^p f(A^p).$$

- (ii) By part (i),  $h(t) = t^p f(t^p)$  is operator monotone on  $(0, \infty)$ . By [8, Corollary 2.6)],  $th(t)^{-1}$  is operator monotone, hence

$$B^{1-p} f(B^p)^{-1} \leq A^{1-p} f(A^p)^{-1}.$$

Therefore

$$A^{p-1} f(A^p) \leq B^{p-1} f(B^p).$$

$\square$

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